Countable and uncountable boundaries in chaotic scattering

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We study the topological structure of basin boundaries of open chaotic Hamiltonian systems in general. We show that basin boundaries can be classified as either type I or type II, according to their topology. Let B be the intersection of the boundary with a one-dimensional curve. In type I boundaries, B is a Cantor set, whereas in type II boundaries B is a Cantor set plus a countably infinite set of isolated points. We show that the occurrence of one or the other type of boundary is determined by the topology of the accessible configuration space, and also by the chosen definition of escapes. We show that the basin boundary may undergo a transition from type I to type II, as the system's energy crosses a critical value. We illustrate our results with a two-dimensional scattering system.

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Transient chaos is a common phenomenon in open systems. It is characterized by the presence of a fractal set of unstable orbits in the phase space (the *invariant set*), which causes nearby orbits to behave erratically, and gives rise to the sensitivity of the dynamics to initial conditions that is the characteristic feature of chaos [1]. Transient chaos is found in a multitude of important physical systems, such as scattering systems [2], systems with escapes [3], dissipative systems [4], etc. One of the most important concepts in open systems is that of *escapes*, which correspond to different asymptotic states the system may reach after the transient is through (we include attractors in this definition). There are usually many possible definitions of escape for any given system. For each escape, its *basin* is defined as the set of points in phase space that evolves to that escape. In two-dimensional scattering, for example, one might associate escape 1 with all orbits with scattering angle in the interval between 0 and π, and escape 2 with those with scattering angle between π and 2π. In this paper, the escapes are defined by a set of surfaces on the configuration space, and an initial condition belongs to the basin of a particular escape if its orbit crosses the corresponding surface for the last time before exiting the scattering region toward infinity. In the above example, the escapes 1 and 2 correspond to the two halves of a circle with a large radius (the radius has to be large enough so that the potential at all points on the circle is negligible). Of particular interest is the geometric structure of the boundary that separates the different escape basins, which is of great importance, both theoretically and in applications [5]. In particular, for three or more escapes, boundaries may have the so-called *Wada property*, which means that every neighborhood of any boundary point contains points belonging to at least three different basins [6]. There have been many studies concerning the structure of basin boundaries for many particular systems and particular definitions of escapes. The important question of how the structure of the invariant set and the choice of the escapes determine the geometry of the basin boundary, however, has not been fully addressed until now. In this article, we study in a general fashion the geometry of basin boundaries for chaotic Hamiltonian systems, how it is affected by the choice of escapes (that is, the choice of the surfaces on the configuration space that define the escapes), and how it reflects the fractal structure of the invariant set. We find that basin boundaries of typical chaotic systems can be classified as one of two types, according to their topology; we refer to them as type I and type II boundaries. For convenience, we define the set B to be the intersection of the boundary with a one-dimensional manifold in phase space. In type I boundaries, B is a Cantor set, consisting of points on the stable manifold of the invariant set; each point in B is the accumulation of other points in the set, and there is no isolated point. In this case, the boundary is everywhere nonsmooth, and every blowup of neighborhoods of any boundary point reveals further structure. In type II boundaries, the set B consists of two components, which we call B_c and B_s. One of the components (B_c) is a Cantor set corresponding to the intersection of the one-dimensional curve used in the definition of B with the stable manifold of the invariant set. The other component (B_s) is a countable (infinite) set of isolated points with limit set on B_c. Contrary to type I boundaries, type II boundaries have smooth parts, corresponding to B_s. Moreover, there are points belonging to B_s arbitrarily close to every point in B_c, and therefore blowups of any neighborhood of a boundary point will always contain smooth parts; the smooth and fractal parts of the boundary are mixed at all scales. We show that for a given Hamiltonian system, the occurrence of one or the other type of boundary depends on the presence or absence of forbidden regions in the configuration space. In other words, the topology of the boundary depends on the topology of the configuration space. More precisely, we show that if for a given energy there is no forbidden region (and therefore the whole configuration space is accessible to the system), then the boundary is always of type II, for any choice of escapes. A necessary (but not sufficient) condition for the occurrence of type I boundaries is the presence of forbidden regions. In this case, we show that if the surfaces defining distinct escapes intersect only within the forbidden regions, then the boundary is always of type I. If any two such surfaces intersect in the accessible region, then the boundary is usually of type II (although it can also be of type I in some cases). We illustrate our findings with a two-dimensional scattering system. We show that this system presents a topological bifurcation (metamorphosis) in which the boundary changes from type II to type I as the energy of the incident particle crosses a critical value. We argue that this phenomenon is general, and is expected to be found in many other systems.

We now introduce a specific system to better explain our results. We choose a system consisting of a particle moving on a two-dimensional plane under a potential V. The potential is taken to be the superposition of three identical "hills":

$$V(x,y)=\sum_{i=1}^{3} V_0(x-x_i,y-y_i), \tag{1}$$

with each "hill" V_0 being spherically symmetric, and having a repulsive core surrounded by an attractive region. Our results do not depend on the particular form of V_0. For definiteness, we use the following form for V_0:

$$V_0(x,y)=A_+e^{-r^2/2\sigma_+^2}-A_-e^{-r^2/2\sigma_-^2}, \tag{2}$$

where A_+, A_-, σ_+, and σ_- are positive constants, and $r^2=x^2+y^2$. In our example, we use $A_+=2.1$, $A_-=2$, $\sigma_+=0.25$, $\sigma_-=0.5$. With this choice, V_0 has a narrow central peak with a maximum of $E_m=0.1$ at the origin, surrounded by an attractive well. For particles with energy E less than 0.1 (but greater than 0), there is a forbidden region around the origin, where the particle cannot penetrate. For $E>0.1$, there is no forbidden region. We choose the three "hills" in Eq. (1) to be placed on the vertices of an equilateral triangle: $x_2=-x_1=4$, $y_1=y_2=0$, $x_3=0$, $y_3=4\sqrt{3}$. Since the distance separating the vertices of the triangle is much greater than both σ_+ and σ_-, the potentials of distinct hills essentially do not overlap. This means that for $E<E_m$, there are three forbidden regions around the vertices of the triangle, whereas if $E>E_m$ there is no forbidden region, and the whole plane is accessible to the particle. Since we are interested in the scattering regime, we only consider particles with $E>0$. We have verified that the scattering is chaotic below a critical energy E_{ch}, with $E_{ch}>0.1$. If the hills were purely repulsive, we would have $E_{ch}=0.1$ [7], but because of the attractive part, there is a fractal invariant set even in the absence of forbidden regions.

We now consider escape basins in this system. We use the sides of the triangle (i.e., the segments connecting the centers of the three hills) to define three different escapes (see Fig. 1). We label the sides of the triangle as in Fig. 1, and for every trajectory that enters the triangle, its escape is defined by the label of the last segment it crosses before it exits the scattering region towards infinity. We will now study the structure of the escape basins for $E<E_m$ and $E>E_m$.

We consider the case $E<E_m$ first. In this case, there are three disconnected forbidden regions surrounding each vertex, schematically shown in Fig. 1 by the dashed circles. Consider now any two initial conditions a and b such that they belong to different basins S_1 and S_2 ($S_1\neq S_2$). Now take any path C in the phase space (restricted to the energy surface $E=\text{const}$), connecting a and b. We will show that somewhere on C lies a point belonging to the escape S_3, with $S_3\neq S_1,S_2$. To show this, we follow C, starting at a and going toward b. Consider now the orbits corresponding to

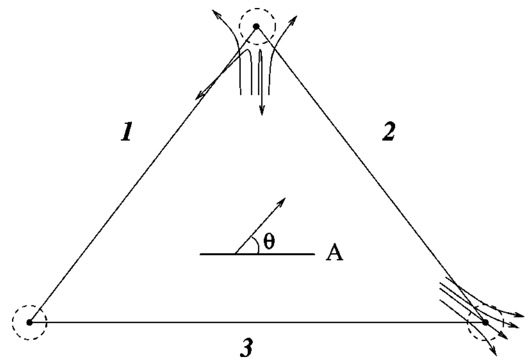

FIG. 1. Schematic representation of the potential (1). The three hills are centered on the vertices of the triangle. The sides 1, 2, and 3 define three escapes. For $E<E_m$, the forbidden regions surrounding each vertex are represented by the dashed circles. The set of trajectories incident on the upper circle illustrates the effect of the forbidden regions on families of orbits. The case of $E>E_m$, when there is no forbidden region, is illustrated by the trajectories on the lower right vertex of the triangle. In the numerical calculations, the initial conditions are picked on the segment A given by $y=\text{const}=2$, $-1.5<x<1.5$, with initial velocity in the direction θ.

the initial conditions on this path. Since the only common point to the two segments corresponding to the escapes S_1 and S_2 lies within a forbidden region F, as we follow C from a to b the orbits will eventually come close to F. Since F is impenetrable to incoming particles, these orbits will suffer large deflections in their trajectories. At some point on the path, the deflection will be large enough to make the orbit leave the scattering region through the third escape S_3. This reasoning is illustrated in Fig. 1 with the set of orbits incident on the upper forbidden region. From the above, we conclude that any neighborhood of every boundary point on C has points belonging to all three escapes. Since the above reasoning is valid for all paths C, we conclude that the boundary has the Wada property. This implies, among other things, that B (the intersection of the basin boundary with C or with any other curve) has no isolated points, because isolated boundary points would have neighborhoods with points belonging to only two basins, contradicting the above. Therefore, B is a Cantor set, and this is a type I boundary. B consists of points that never leave the scattering region. In other words, B is on the stable manifold of the invariant set. Notice that in the above arguments we used the fact that forbidden regions exist, and that the segments defining different escapes intersect only within the forbidden regions, so that orbits cannot go smoothly from one escape to another as we change the initial conditions.

Consider now the case $E>E_m$, with $E<E_{ch}$, so that the scattering is still chaotic, but now there are no forbidden regions, and the whole $\mathbf{R}^2$ plane is accessible to the particles. In this case, one-parameter families of orbits go smoothly from one escape to the other, as depicted schematically in Fig. 1 with the trajectories incident on the lower right forbidden region. In this case, there are isolated boundary points on B, corresponding to orbits that leave the scattering region along trajectories that cross the triangle exactly on one of the vertices. Besides the isolated points, B also includes a Cantor

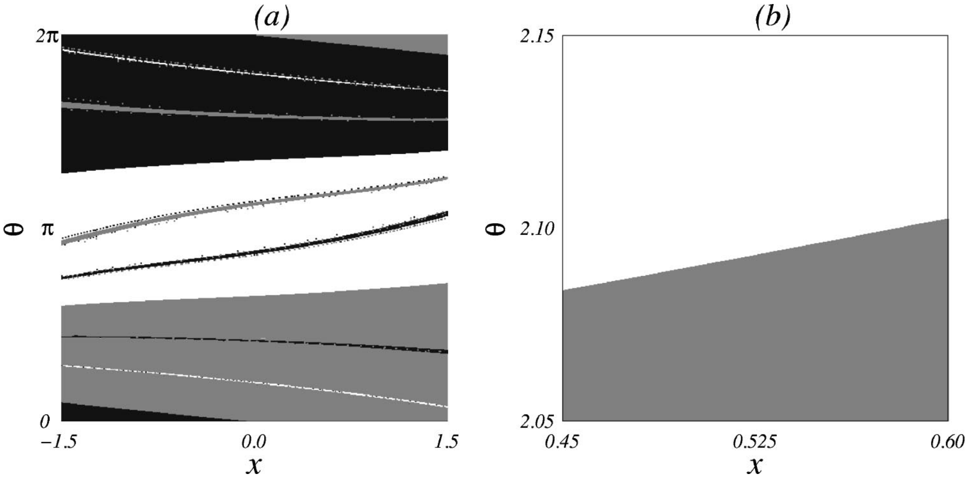

FIG. 2. Picture of the basins corresponding to the three escapes defined in Fig. 1, for $E=0.15>E_m$. The initial conditions are picked as explained in Fig. 1 and in the text, on a 500×500 grid. Points are colored white if they belong to basin 1, gray if they belong to basin 2, and black if they belong to basin 3, in the notation of Fig. 1. (b) is a magnification of a region of (a). The region shown in (b) clearly has a smooth boundary.

set B_c, corresponding to points lying on the stable manifold of the fractal invariant set. Consider now a small neighborhood of a point P on B_c. Points in this neighborhood correspond to orbits that stay a long time within the scattering region, following closely an orbit in the invariant set for a long time, before escaping. Small variations of the initial conditions in this region will typically lead to large changes in the asymptotic state reached by the orbit, and consequently to a different escape. We expect that in any neighborhood of P we can find a path of initial conditions that leads from a basin to another smoothly, as explained above. This means that the isolated boundary points accumulate on the fractal set of points on the stable manifold of the invariant set. We conclude that for $E>E_m$ (and $E<E_{ch}$), the boundary is of type II. The isolated points of the boundary, not present in type I boundaries, owe their existence to the absence of forbidden regions in the configuration space.

From the above discussion, we see that as the energy of incoming particles crosses the critical value E_m, the basin boundary undergoes a topological transition from type II ($E>E_m$) to type I ($E<E_m$). We show that this transition does indeed occur, through a numerical calculation of the basins. We take initial points lying on the segment A shown in Fig. 1, with initial velocities in the direction θ, with $0\leqslant\theta<2\pi$; the magnitude of the velocity is fixed by the conservation of energy. We integrate the equations of motion on a grid of 500×500 points, and we plot the escape as a function of the initial conditions using a different gray scale for each escape, as explained in the caption of Fig. 2. The results for $E=0.15>E_m$ are shown in Fig. 2, and those for $E=0.05<E_m$ are displayed in Fig. 3. In Fig. 2(a), we see that there are regions where the three basins are very intricately intertwined, corresponding to neighborhoods of the fractal part of the boundary, and we also see that there are smooth parts of the boundary where only two basins meet. Figure 2(b) shows a magnification of a smooth part of the boundary. Blowups of other regions show that smooth boundaries are present at all scales, and that they accumulate on the fractal parts of the boundary. In other words, we have a type II boundary, as expected. For $E<E_m$, however, regions of the boundary that were smooth for $E>E_m$ now have a fine structure, with points belonging to all three basins, due to the presence of the forbidden regions, as explained above. Figure 3 shows a magnification of the same region displayed in Fig. 2(b), but for $E=0.05$, and we can see that the boundary is no longer smooth. Further magnifications reveal structure at all scales. Blowups of other regions in Fig. 3 show that there are no smooth boundaries, showing that this is a type I boundary, as it should be. As the energy drops from $E>E_m$ to $E<E_m$, all points of the boundary that were formerly smooth (nonfractal) become fractal, through the sudden creation of new orbits in the invariant set. These new orbits correspond to the particle bouncing between the newly created forbidden regions.

We note that, although we used the system described by Eqs. (1) and (2) to facilitate the discussion and to exemplify our reasoning, the connection between the topology of the basin boundary and the topology of the accessible configuration space uncovered by us is a general one, since our reasoning did not depend on any particular feature of the model. We therefore conclude that, in general, a basin boundary is of type I if the points that separate different escapes lie within forbidden regions; otherwise the boundary is usually of type II (although it can be type I in some special

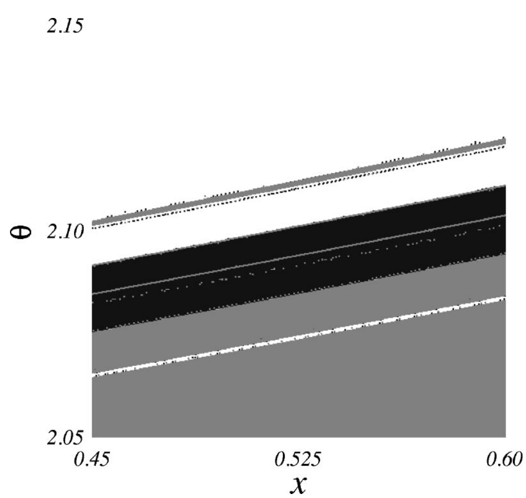

FIG. 3. Same as Fig. 2, with $E=0.05<E_m$. The region shown is the same as in Fig. 2(b), and it has a fractal structure now.

cases). Systems with no forbidden regions (for a given energy) always have type II boundaries, whereas systems having forbidden regions may admit type I boundaries. Notice that, even if forbidden regions are present, the boundary may be type II if the above condition on the definition of escapes is not satisfied. For instance, in the system (1), if the sides of the triangle in Fig. 1 are enlarged so that the vertices lie outside the forbidden regions, we will have a type II boundary, even for $E < E_m$. We have verified numerically that this is indeed the case.

Most of the basin boundaries studied in the literature are type I. Type II boundaries have been found in incompressible two-dimensional flow [8], in maps [9], and in three-dimensional scattering [10], discussed below.

We now make some final remarks. Type II basins cannot have the Wada property, because they have smooth components (although the fractal component of the basin may be Wada). This means that systems can only have Wada basins if there are forbidden regions in the configuration space (but this is not a sufficient condition). In the example presented above, the type II–type I transition is also a non-Wada–Wada transition. An important and widely used definition of escapes in scattering systems is that in which each escape corresponds to a range of the scattering angle ϕ, such as the example mentioned in the beginning of the paper. Each scattering angle is associated with a point on a very large circle centered on the scattering region. Each range of ϕ is thus represented by an arc of this circle. Since the circle has a large radius (it lies on the asymptotically free region, where the potential can be neglected), it is very far away from any forbidden region. We conclude that basins defined by ranges of the scattering angle are always type II basins (and consequently non-Wada basins). Finally, although we have restricted ourselves to systems with two degrees of freedom, this choice was only made for convenience, and our arguments hold for higher dimensions also. In three-dimensional potentials, for example, escapes are defined by two-dimensional surfaces, and the condition for the occurrence of type I basins is that the one-dimensional intersections of those surfaces have to lie within forbidden regions. The basin transition studied in [10] is actually a type I–type II transition, although this terminology was not used there. This transition has a very different origin from the one we presented here, and is a purely three-dimensional phenomenon.

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